FRACTIONAL CALCULUS AND ITS APPLICATIONS TO THE NON-HOMOGENEOUS GAUSS' EQUATIONS

K. Nishimoto
College of Engineering
Koriyama, Fukushima-ken
Japan

ABSTRACT

Some other many papers on fractional calculus have been reported by the author. In this paper, firstly, we will show a table of fractional diferintegrations of elementary functions which is obtained obeying the definition of author.

Next we will show an application of our fractional calculus to the non-homogeneous Gauss' equation which is a differential equation of Pacht's type.

RESUMEN

El autor ha publicado varios trabajos sobre el cálculo fraccional. En este trabajo primero, se da una tabla de diferintegraciones de algunas funciones elementales, usando la definición del autor. Además se resuelve la ecuación diferencial no-homogénea de Gauss mediante la aplicación del cálculo fraccional.

60. INTRODUCTION
(Definition of fractional calculus)

Definition. If f(z) is a regular function and it has no branch point inside C and on Cc = \{C, C', \}, C is an integral curve along the cut joining two points z and -w = \text{Im}(z), and C' is an integral curve along the cut joining two points z and +w = \text{Im}(z),

\[ f_{\nu} = f_{\nu}(z) \frac{\Gamma(n+1)}{\Gamma(n+1)} \int_{C} \frac{f(z)}{(z-z')^{n+1}} dz' \]

\[ \Gamma \text{ : Gamma function} \]
\[ \nu \in \mathbb{C} \]
and

where \( C \neq z \), \(-\arg(C-z)\leq \pi \) for \( C \) and \( 0 \leq \arg(C-z) \leq \pi \) for \( C' \), then \( f_{\nu}(z) \) is the fractional derivative of order \( \nu \) and \( f_{\nu}(w) \) is the fractional integral of order \( -\nu \), if \( f_{\nu}(z) \) exists (consider the principal value of \( f \) for many valued function).

61. TABLE OF FRACTIONAL DIFERTEGRATIONS OF ELEMENTARY FUNCTIONS AND SOME LEMMAS

Through the author's definition for fractional differintegration, we have Table 1. And, to make sure, we will show List 1. [1] [2] [9]

Lemma 1. Let \( f(z) \) be regular and one valued functions. If \( f_{\nu}(0) \) and \( f_{\nu}(0) \) exist, then

\[ (f_{\nu})_{\nu} = f_{\nu} - f_{\nu+\nu} \] (index law) \hspace{1cm} (1)

Lemma 2. Let \( u(z) \) and \( v(z) \) be regular and one valued functions. If \( u_{\nu} \) and \( v_{\nu} \) exist, then

\[ (u \cdot v)_{\nu} = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1)} - \frac{\Gamma(n+1)}{\Gamma(n+1)} u_{\nu-n} v_{n} \] \hspace{1cm} (2)

Refer [6], [10] and [21] for these Lemmas.
Table 1. Nishihara's Fractional differintegration of elementary functions

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( f(a\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 )</td>
<td>( x )</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>( (\alpha+1)/x )</td>
</tr>
</tbody>
</table>

1. \( \cos (x+a) \)  
2. \( \sin (x+a) \)  
3. \( \cosh x \)  
4. \( \sinh x \)  
5. \( \cos \sinh x \)  
6. \( \sin \cosh x \)  
7. \( \cosh (x+a) \)  
8. \( \sin (x+a) \)  
9. \( \exp (x+a) \)  
10. \( \exp (-x) \)  
11. \( \exp (-x) \)  
12. \( \exp (-x) \)  

(See the footnotes of List 1.)

**List 1.**

**Functions**

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( c )</th>
<th>( \alpha )</th>
<th>( \gamma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>( (\alpha+1)/x )</td>
<td>( (\alpha+1)/x )</td>
<td>( \cos \alpha )</td>
</tr>
</tbody>
</table>

- **Integral**
  - For \( \Re > 0 \)
  - For \( \Re = 0 \)
  - For \( \Re < 0 \)

- **Derivative**
  - For \( \Re > 0 \)
  - For \( \Re = 0 \)
  - For \( \Re < 0 \)

**Note 1.** \( 2^\alpha \): set of the positive integers, \( 2^\alpha \): set of the negative integers.

**Note 2.** \( \log x \) diverge for \( \Re + x \) by the direct calculation (following author's definition). However we calculate as \( (1-x)^{\alpha}/x \).

**Note 3.** In case of \( x \in \mathbb{E} \) and \( x \in \mathbb{E} \), calculate as for example \( a=2, x=-1 \), \( \text{Log}(1-x)^{\alpha}/x = \text{Log}(1-x)^{\alpha}/x \).

---

6.2 SOLUTIONS TO HOMOGENEOUS AND
NON-HOMOGENEOUS HYPERGEOMETRIC
EQUATIONS OF GAUSS

\[ (a + 1)x + y = x \quad (x \neq 0,1) \]  
(1)

The homogeneous hypergeometric differential equation

\[
\frac{d^2y}{dx^2} + \left( \frac{a + \alpha + \beta}{x} \right) \frac{dy}{dx} + \left( \frac{-a + \gamma}{x} \right) y = 0
\]

has a particular solution of the form

\[
w(x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k (k!)} \left( \frac{x}{x-1} \right)^k
\]
(2)

where \( \alpha = \alpha (x) \), \( \beta = \beta (x) \), and \( \alpha, \beta \) and \( \gamma \) are constants.

Proof. Setting \( y = w(x) \)

yields \( y = w(x) \) (by Lemma 1.)
(3)

and \( y = w(x) \),
(4)

where \( w = w(x) \)

Substituting (1), (4) and (5) into (1), we obtain

\[
w(x)^2 = w(x) + w(x) = w(x) \left( \frac{(x+\alpha) (x+\beta)}{(x+\gamma)} \right) = \frac{1}{(x+1)} \frac{1}{(x+1)}
\]
(6)

that is

\[
(w(x)^2) = (w(x)) (w(x)) \left( \frac{(x+\alpha) (x+\beta)}{(x+\gamma)} \right) \left( \frac{(x+\alpha) (x+\beta)}{(x+\gamma)} \right) = f(x)
\]
(7)

Consequently we have

\[
w(x)^2 = w(x) \left( \frac{(x+\alpha) (x+\beta)}{(x+\gamma)} \right) \left( \frac{(x+\alpha) (x+\beta)}{(x+\gamma)} \right) = f(x)
\]
(8)

from (7), since

\[
\frac{(n-z)^n}{n} = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k (k!)} \left( \frac{x}{x-1} \right)^k
\]
(9)

where \( n \in \mathbb{N} \), \( \alpha, \beta \) and \( \gamma \) are constants.

Put \( u = \frac{x}{x-1} \)
(10)

In (8), we have then

\[
u_1 = u
\]
(11)

A particular solution of this linear differential equation of first order is given by

\[
u_1 = \left( \frac{x}{x-1} \right) (x-1) \left( \frac{x}{x-1} \right) = \frac{x}{x-1} \left( \frac{x}{x-1} \right)
\]
(12)

Therefore we obtain, using (10) and (3),

\[
u = \nu_0 = (n-x) \left( \frac{x}{x-1} \right)
\]
(13)

and

\[
u = \nu_0 = (n-x) \left( \frac{x}{x-1} \right)
\]

as a particular solution to the equation (1).

Inversely we have

\[
u_1 = \frac{n}{x-1}
\]
(14)

and

\[
u = \frac{n}{x-1}
\]

from (13). Substituting (13), (15) and (16) into the left hand side of (1), we obtain

\[
L.H.S. of (1) = (n-x) \left( \frac{x}{x-1} \right) + (n-x) \left( \frac{x}{x-1} \right)
\]
(15)

from (7), since

\[
-x-97
\]

Proof. Putting \( \phi = \psi_1 \) \( (24) \)
in \( (22) \), we have then (see the proof in \((1)\))
\[ w_2(z^2-x^2) + \psi_1 \left[ z(1+z) + (\alpha-\gamma) \right] = 0 \] \( (25) \)

A particular solution of this equation is given as follows.
\[ w = (z^{\alpha-\gamma}(z-1)^{-\beta-1})_{\alpha-1} \] \( (26) \)

Substituting \((26)\) into \((24)\), we have then
\[ \phi = \psi_1 = (z^{\alpha-\gamma}(z-1)^{-\beta-1})_{\alpha-1} \] \( (27) \)

Inversely we obtain
\[ \psi_1 = \psi_{1+1} \] \( (28) \)
and
\[ \psi_2 = \psi_{1+1} \] \( (29) \)

Substituting \((27)\), \((28)\) and \((29)\) into the left hand side of \((22)\), we have then
\[ \text{L.H.S. of } (22) = \left( (w_2(z^2-x^2) + w_1 \left[ z(1+z) + (\alpha-\gamma) \right] \right)_{\alpha-1} = 0 \] \( (30) \)

has a solution of the form
\[ \phi = (z^{\alpha-\gamma}(z-1)^{-\beta-1})_{\alpha-1} \] \( (31) \)

where \( \phi = \phi(z) \) and \( z \in \mathbb{C} \). The equation \((22)\) is hypergeometric differential equation of Gauss \([1]\).

- 98 -

Changing \( a \) and \( \beta \) in (27), we have other solution

\[
\phi = (z^{-\alpha}(z-1)^{-\gamma-1})_{\alpha-1},
\]

(35)

if \( \alpha \neq \beta \).

(III) Theorem 3. If \( f(0) \neq 0 \) exists, then the fractional differential equation

\[
\phi = (f(0)^{z^{-\alpha}(z-1)^{-\gamma}})_{\alpha-1} (z^{-\gamma}(z-1)^{-\beta-1})_{\beta-1} +
\]

satisfies the differential equation of Puchs type (1), where \( \gamma \in \mathbb{C} \).

Proof. It is clear by the Theorems 1 and 2.

8.3 TRUE COLORS OF GAUSS' HYPERGEOMETRIC FUNCTIONS

Theorem 4. We have

\[
F(\beta-\gamma+1, \beta; \beta-\alpha+1; 1/z)
\]

\[
e^{-1\Gamma(\gamma) \Gamma(\beta+1) \Gamma(\beta-\alpha+1) / \Gamma(\beta)} z^{\beta}(z^{-\alpha}(z-1)^{-\gamma-1})_{\alpha-1}
\]

(1)

for \( |z| > 1 \), and

(II) \( F(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; z) \)

\[
e^{\Gamma(\alpha+\beta-\gamma) \Gamma(\alpha+1) \Gamma(\alpha-\gamma) (z^{-\alpha}(z-1)^{-\gamma-1})_{\alpha-1}
\]

(2)

for \( |z| < 1 \).

Proof of (i). We have

\[
(z-1) = e^{\Gamma(1-z)}
\]

(3)

\[
(z^{-\alpha}(z-1)^{-\gamma-1})_{\alpha-1}
\]

(4)

hence we obtain

\[
F(\beta-\gamma+1, \beta; \beta-\alpha+1; 1/z)
\]

\[
e^{-1\Gamma(\gamma) \Gamma(\beta+1) \Gamma(\beta-\alpha+1) / \Gamma(\beta)} z^{\beta}(z^{-\alpha}(z-1)^{-\gamma-1})_{\alpha-1}
\]

(5)

for \( |z| > 1 \). Therefore we have (i) from above result (7).

Proof of (ii). We have

\[
e^{\Gamma(1-z)}
\]

(6)

\[
(z^{-\alpha}(z-1)^{-\gamma-1})_{\alpha-1}
\]

(7)

hence we obtain

\[
F(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; z)
\]

\[
e^{\Gamma(\alpha+\beta-\gamma) \Gamma(\alpha+1) \Gamma(\alpha-\gamma) (z^{-\alpha}(z-1)^{-\gamma-1})_{\alpha-1}
\]

(8)

for \( |z| < 1 \).

Proof of (i). We have

\[
(z^{-\alpha}(z-1)^{-\gamma-1})_{\alpha-1}
\]

(9)
The differential equation is given as
\[ \psi = \left( \left( \frac{1}{x-1} \right) \cdot \frac{1}{x} \right) \cdot \frac{x-1}{x} \cdot \frac{y_{2}}{x} \] (6)

for \(|x| < 1\). We have (11) from (11).

4. SOME EXAMPLES

Putting \(a = v, \beta = v - 1, \gamma = v + 1\), we have

\[ \phi_2 \left( (y-a)/k \right) + \phi_1 \left( (2y-a)/k \right) + \phi \left( \psi \right) = f \] (1)

and

\[ \psi = \left( \left( \frac{1}{x-1} \right) \cdot \frac{1}{x} \right) \cdot \frac{x-1}{x} \cdot \frac{y_{2}}{x} \] (2)

from the Theorem 1, and

\[ \phi_2 \left( (y-a)/k \right) + \phi_1 \left( (2y-a)/k \right) + \phi \left( \psi \right) = 0 \] (3)

and

\[ \psi = (z - \log z) \cdot \psi \] (4)

from the Theorem 2, respectively [15].

(1) More practically, let \( \psi = \frac{1}{2} \) and

\[ f = x^{-1/2} \], we have then

\[ \phi_2 \left( (y-a)/k \right) + \phi_1 \left( (2y-a)/k \right) + \phi \left( \frac{1}{2} \right) = x^{1/2} \] (5)

and

\[ f = x^{-1/2} \], we have the following

\[ \psi_1 \left( \psi \right) = \psi_1 \left( \psi \right) \] (12)

hence

\[ \psi_1 \left( \psi \right) = \psi_2 \] (13)

and

\[ \psi_2 = \psi_2 \] (14)

- 100 -
Substituting (12), (13) and (14) into the left hand side of (5), we have then

\[ L.H.S. \text{ of (5)} = w \cdot (z^2-z) - w \cdot \left( z + \frac{1}{2} \right) + w \cdot \left( \frac{3}{4} \right) \]

(15)

\[ = (w \cdot z^2) - \left( w \cdot z \right) \]

(16)

\[ - (w \cdot z^2) \cdot \left( \frac{3}{4} \right) \]

(17)

\[ = \frac{f \cdot \left( \frac{3}{4} \right)}{\gamma} \]

(18)

\[ = \frac{e - \gamma}{\gamma} \]

(19)

(ii) Next, we have

\[ \phi_1 = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{n} G(k) \left( \frac{1}{2} - k \right) z^{k-\left( \frac{1}{2} \right)} \]

(20)

and

\[ \phi_2 = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{n} G(k) \left( \frac{1}{2} - k \right) \left( - \frac{1}{2} - k \right) z^{k-\left( \frac{1}{2} \right)} \]

(21)

from (7).

Substituting (7), (20) and (21) into the left hand side of (5), we have then

\[ L.H.S. \text{ of (5)} = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{n} G(k) \left( k^2 - k \right) z^{k-\left( \frac{1}{2} \right)} - k \]

(22)

\[ - \frac{1}{\sqrt{\pi}} \sum_{k=1}^{n} G(k) \left( k^2 - k \right) z^{k-\left( \frac{1}{2} \right)} - k \]

(23)

\[ = e \cdot \gamma \]

(24)

since

\[ G(1) = \sqrt{\pi}/2 \]

and

\[ C(k \cdot l) \cdot \left( (k \cdot l)^2 + (k \cdot l) \right) = G(k) \left( k^2 - \frac{1}{2} \right) \]

(25)

again

(iii) Let \( \nu = -1/2 \) in (3), we have then

\[ \phi_2 \cdot (z^2-z) - \phi_1 \cdot (z + \frac{1}{2}) + \phi \cdot \frac{3}{4} = 0 \]

(26)

A particular solution to this Gauss’ equation is given as

\[ \phi = \left( z - \log z \right) \gamma \]

(27)

by (4). Inversely we have

\[ \phi_1 = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{n} G(k) \left( \frac{1}{2} - k \right) \left( - \frac{1}{2} - k \right) z^{k-\left( \frac{1}{2} \right)} \]

(28)

from (28).

Substituting (28), (29) and (30) into the left hand side of (26), we obtain

\[ \frac{1}{\sqrt{\pi}} \sum_{k=1}^{n} G(k) \left( k^2 - k \right) z^{k-\left( \frac{1}{2} \right)} - k \]

(31)

However, the solution (34), in its form, is a more desirable and wide one than (36).

Let \( v = 2 \) in (3) and (4), we have then

\[
\phi_1(x^2 - z) + \phi_1(4z - 3) + 2 = 0 \quad (z \neq 0, 1)
\]

and

\[
\phi = (z - \log z)_2 = z^2
\]

respectively.

By assuming \( \phi = (x)_2 \), the second solution independent to the first particular solution (33) is given by

\[
\phi = x^{-1} - x^{-2} \log(1-x) \quad (z \neq 0, 1).
\]

On the other hand, assuming a solution of the form of power series, we have then

\[
\phi = F(n, v - 1; v + 1; z)
\]

as a particular solution to the equation (3), where

\[
F(a, b; y; z) \text{ is the Hypergeometric function.}
\]

Put \( v = 2 \) in (35), we have

\[
\phi = 2 \sum_{n=0}^{\infty} \frac{z^n}{2n} \quad (|z| < 1)
\]

as a solution to the equation (32).

Solution (34) coincides with (36) except the coefficient of constant, since

\[
\log(1 - z) = - \sum_{n=1}^{\infty} \frac{z^n}{n} \quad (|z| < 1).
\]

If \( n = 0 \), (38) and (39) are reduced to

\[
\phi_2(x^2 - z) = \phi_1 + 2z + \phi_2 = 0 \quad (z \neq 0, 1)
\]

and

\[
\phi = (z - \log z)_3
\]

respectively. Clearly, (41) satisfies equation (40).

And in case of \( n = 1 \), for example, (38) and (39) are reduced to

\[
\phi_2(x^2 - z) - \phi_1 + 2z + \phi_2 = 0 \quad (z \neq 0, 1)
\]

and to

\[
\phi = (z - \log z)_4
\]

respectively. Hence we have

\[
\phi_1 = z - \log z
\]

and

\[
\phi_2 = 1 - z^{-1}
\]

from (43). Substituting (43), (44) and (45) into the left hand side of (62), we have then

- 102 -

L.H.S. of (42) $= 1 \neq 0$.  
(46)

That is, the function (43) does not satisfy the differential equation (42).

However, if we adopt

$$
\phi = (z - \log z)^{-1}
$$

and

$$
= / \ (z - \log z) \ dz
$$

as a solution (c is an arbitrary constant of integration) to the equation (42) and substitute this into the L.H.S. of (42), we obtain

L.H.S. of (42) $= 1 + 2c$.  
(48)

Consequently, determining $c = -1/2$ in (47), we obtain

$$
\phi = \frac{1}{2} z^2 - (z \log z - z) - \frac{1}{2}
$$

as a solution to the equation (42). And (49) satisfies (42) clearly.

REFERENCES


Acknowledgement

The author appreciates to Professor S.L. Kalla, who invited the author for author's contribution to the proceeding "Revista Técnica (Special Number in commemoration of its 15th anniversary)", for this goodwill and kindness.


[15] NISHIMOTO, K.: "An Application of Fractional Calculus to a Differential Equation of Fuchs Type \( \phi_n(z, 0; \phi; z, 1) \) and \( \psi_n(z, 0; \phi; z, 1) \)." J. Coll. Engng. Nihon Univ., B-28 (1987), 9-11.

[16] NISHIMOTO, K.: "Application of Fractional Calculus to a Differential Equation of Fuchs Type \( \phi_n(z, 0; \phi; z, 1) \) and \( \psi_n(z, 0; \phi; z, 1) \)." J. Coll. Engng. Nihon Univ., B-27 (1986), 5-16.


[18] NISHIMOTO, K.: "An Application of Fractional Calculus to the Differential Equation of Fuchs Type \( \phi_n(z, 0; \phi; z, 1) \) and \( \psi_n(z, 0; \phi; z, 1) \)." J. Coll. Engng. Nihon Univ., B-28 (1987), 9-11.


Recibido el 11 de marzo de 1987.

---